$$e+r=4p^2\left(1-\frac{4}{3}p^2\right)\left(1-\frac{7}{3}p^2\right)^{-1}\left(\sqrt{1-\frac{4}{3}p^2}-p\right)^2.$$

There follows from the boundary condition (3.1)

$$\ln\frac{r}{a}=-2Mq\sqrt{1-\frac{4}{3}p^2}\left(1-\frac{7}{3}p^2\right)^{-1}.$$

The normal pressure on the wedge face is expressed in the form

$$\sigma_{22}=-\frac{k}{p^2}+\frac{k}{p^2}\left(1-\frac{7}{3}p^2\right)\ln\frac{r}{a}+o\left(\frac{1}{M}\right).$$

The pressure change on the wedge face is shown in Fig. 3 for $p^2 = 0.3$ and $q = 0.1$ as a function of the velocity of wedge penetration. We obtain the minimal pressure in the elastic solution for $M^2 = 1.18$. As M decreases and increases the pressure rises. At the value $M^2 = 99.82$ plastic flow sets in at the wedge face (the domain bOc in Fig. 1) and the pressure growth is terminated. For $M^2 = 100.20$ plasticity sets in even in the domain aOb. Later the pressure grows in proportion to M as M increases.

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THEORY OF IDEAL PLASTICITY OF MULTICOMPONENT MIXTURES

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1. We consider a rigidly plastic micro-inhomogeneous isotropic medium consisting of n different components interconnected by ideal adhesion. Let the plastic properties of each component be described by the surface flow taking the hydrostatic pressure into account

$$s_{ij}s_{ij}+a_s\sigma_{pp}^2=k_s^2,\quad s=1,2,\ldots,n,$$

where $s_{ij} = \sigma_{ij} - \delta_{ij}\sigma_{pp}/3$, σ_{ij} is the stress tensor, k_s are the component yield points, and a_s are parameters characterizing their volume compressibility.

The structure of such a medium can be described by a system of random indicator functions of the coordinates $\varkappa_1(\mathbf{r}), \varkappa_2(\mathbf{r}), \ldots, \varkappa_n(\mathbf{r})$, from which each function $\varkappa_s(\mathbf{r})$ equals unity on a set of points of the s-th component and equals zero outside this set. By using these functions the local associated flow law of the composite material under consideration can be written in the form [1]

$$\sigma_{ij}(\mathbf{r})=k(\mathbf{r})\frac{\varepsilon_{ij}(\mathbf{r})-\delta_{ij}b(\mathbf{r})\varepsilon_{pp}(\mathbf{r})}{\sqrt{\varepsilon_{kl}(\mathbf{r})\varepsilon_{kl}(\mathbf{r})-b(\mathbf{r})\varepsilon_{pp}^2(\mathbf{r})}},\tag{1.1}$$

where $\varepsilon_{ij}(\mathbf{r})$ is the strain rate tensor; $k(\mathbf{r})=\sum_{s=1}^{n}k_s\varkappa_s(\mathbf{r})$; and

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$$b(\mathbf{r})=\sum_{s=1} b_s\varkappa_s(\mathbf{r});\quad b_s=(3a_s-1)/9a_s.$$

All the functions $\varkappa_s(\mathbf{r})$, the stress and strain rate tensors are assumed statistically homogeneous and ergodic fields, and their mathematical expectations agree with the components averaged over the volumes V_s and over the total volume of the medium $V = V_1 + V_2 + \ldots + V_n$ [2]:

$$\langle(\ldots)\rangle=\frac{1}{V}\int\limits_V(\ldots)\,d\mathbf{r},\quad \langle(\ldots)\rangle_s=\frac{1}{V_s}\int\limits_{V_s}(\ldots)\,d\mathbf{r},\quad s=1,2,\ldots,n.$$

We replace the dissipative function by its mean value $D = \langle\sigma_{ij}\varepsilon_{ij}\rangle$ in the total volume V in (1.1) for the associated flow law of the medium. This assumption linearizes the relationship between the local stochastic stress and strain rate fields, and the relationship (1.1) takes the form [1]:

$$D\sigma_{ij}(\mathbf{r})=k^2(\mathbf{r})\varepsilon_{ij}(\mathbf{r})-k^2(\mathbf{r})b(\mathbf{r})\delta_{ij}\varepsilon_{pp}(\mathbf{r}). \tag{1.2}$$

Substituting the local governing equations (1.2) into the equilibrium equation $\sigma_{ij,j} = 0$, we obtain

$$k_1^2\varepsilon'_{ij,j}-k_1^2b_1\varepsilon'_{pp,i}-\tau'_{ij,j}=0,$$
$$-\tau_{ij}=\sum_{s=1}^n\left[k_s^2\right]\varkappa_s\varepsilon_{ij}-\delta_{ij}\sum_{s=1}^n\left[k_s^2b_s\right]\varkappa_s\varepsilon_{pp},\ \left[k_s^2\right]=k_s^2-k_1^2, \tag{1.3}$$

$[k_s^2b_s] = k_s^2b_s - k_1^2b_1$, the primes denote fluctuation of the quantities in the volume V. Appending the Cauchy formula $2\varepsilon_{ij} = v_{i,j} + v_{j,i}$, relating the displacement velocity vector components $v_i(\mathbf{r})$ to the strain rate tensor components, to (1.3), we obtain a closed system of equations for the flow of a multicomponent mixture for which the boundary conditions are the conditions of no fluctuations of quantities on the surface of the volume V. We introduce the Green's tensor

$$G_{ik}^{(1)}(\mathbf{r})=\frac{1}{8\pi k_1^2}\left(\delta_{ik}r_{,pp}-\frac{1-b_1}{2-b_1}r_{,ik}\right),\quad r=|\mathbf{r}|,$$

which is used to reduce the equilibrium equation of the medium to a system of integral equations [1]

$$\varepsilon'_{ij}(\mathbf{r})=\int\limits_V G^{(1)}_{i(k,l)j}(\mathbf{r}-\mathbf{r}_1)\,\tau'_{kl}(\mathbf{r}_1)\,d\mathbf{r}_1. \tag{1.4}$$

To determine the effective constants of the multicomponent mixture, a relationship must be established between the macroscopic stresses and strain rates. Let us take the average of the relationship (1.2) over the whole body volume V and let us apply the rule of mechanical mixing of the phases

$$D\langle\sigma_{ij}\rangle=\sum_{s=1}^n k_s^2c_s\left(\langle\varepsilon_{ij}\rangle_s-\delta_{ij}b_s\langle\varepsilon_{pp}\rangle_s\right), \tag{1.5}$$

where $c_s = V_sV^{-1}$ are the bulk contents of the components. Equations (1.5) show that to establish a macroscopic rheological law it is required to calculate the strain rate averaged over the component volumes and the mean energy dissipation density D. The quantities $\langle\varepsilon_{ij}\rangle_s$ can be found from the relationships [3]

$$\langle\varepsilon_{ij}\rangle_s=\langle\varepsilon_{ij}\rangle+c_s^{-1}\langle\varkappa'_s\varepsilon'_{ij}\rangle. \tag{1.6}$$

Let us calculate the quantity $\langle\varkappa'_s\varepsilon'_{ij}\rangle$. We multiply (1.4) by $\varkappa'_s(\mathbf{r})$ and take the average over the whole volume V:

$$\langle\varkappa'_s\varepsilon'_{ij}\rangle=\int\limits_V G^{(1)}_{i(k,l)j}(\mathbf{r}_1)\,\langle\varkappa'_s(\mathbf{r})\,\tau'_{kl}(\mathbf{r}+\mathbf{r}_1)\rangle\,d\mathbf{r}_1.$$

In evaluating the integral on the right side, we limit ourselves to a singular approximation by omitting the formal parts of the second derivatives of the Green's tensor; then [3, 4]

$$\langle\varkappa'_s\varepsilon'_{ij}\rangle=(\alpha_1I_{ijkl}-\beta_1\delta_{ij}\delta_{kl})\,\langle\varkappa'_s\tau_{kl}\rangle/k_1^2. \tag{1.7}$$

Here $\alpha_1 = 2(4-3b_1)/15(1-b_1)$; $\beta_1 = (1-2b_1)/15(1-b_1)$; I_{ijkl} is the unit tensor. Substituting the expression for τ_{ij} into (1.7) and extracting the volume and deviator parts, we find

$$\langle\varkappa'_qe'_{ij}\rangle=-\alpha_1\sum_{s=1}^n\left[k_s^2\right]\langle\varkappa'_q\varkappa_se_{ij}\rangle/k_1^2, \tag{1.8}$$

$$\langle \varkappa_q' \varepsilon_{pp}' \rangle = -(\alpha_1 - 3\beta_1) \sum_{s=1}^{n} \left([k_s^2] - 3 [k_s^2 b_s] \right) \langle \varkappa_q' \varkappa_s \varepsilon_{pp} \rangle / k_1^2,$$

where $e_{ij} = \varepsilon_{ij} - (1/3)\delta_{ij}\varepsilon_{pp}$. Eliminating the quantity $\langle \varkappa_q' \varepsilon_{ij}' \rangle$ from (1.6) and (1.8) and taking into account the relationships

$$\langle \varkappa_q' \varkappa_s' f \rangle = \begin{cases} c_q c_s (\langle f \rangle - \langle f \rangle_q - \langle f \rangle_s), & q \neq s, \\ c_s (c_s \langle f \rangle + (1 - 2c_s) \langle f \rangle_s), & q = s, \end{cases}$$

$$\langle \varkappa_q' \varkappa_s' \rangle = \begin{cases} -c_q c_s, & q \neq s, \\ c_s (1 - c_s), & q = s, \end{cases}$$

we express the strain rate components averaged over the volume in terms of the macroscopic quantities

$$\langle e_{ij} \rangle_q = \frac{\alpha_1 D \langle s_{ij} \rangle + (1 - \alpha_1) k_1^2 \langle e_{ij} \rangle}{k_1^2 + \alpha_1 [k_q^2]},$$

$$\langle \varepsilon_{pp} \rangle_q = \frac{\gamma_1 D \langle \sigma_{pp} \rangle + k_1^2 (1 - 3b_1)(1 - \gamma_1) \langle \varepsilon_{pp} \rangle}{k_1^2 (1 - 3b_1) + \gamma_1 \left([k_q^2] - 3 [k_q^2 b_q] \right)}, \tag{1.9}$$

$$\gamma_1 = (3b_1 - 1)/3(b_1 - 1).$$

Substituting (1.9) into (1.5) and separating it into volume and deviator parts, we obtain

$$D \langle s_{ij} \rangle = k_1^2 \frac{(1 - \alpha_1) A_1}{1 - \alpha_1 A_1} \langle e_{ij} \rangle, \quad D \langle \sigma_{pp} \rangle = k_1^2 (1 - 3b_1) \frac{(1 - \gamma_1) B_1}{1 - \gamma_1 B_1} \langle \varepsilon_{pp} \rangle. \tag{1.10}$$

Here

$$A_1 = \sum_{s=1}^{n} \frac{k_s^2 c_s}{k_1^2 + \alpha_1 [k_s^2]}; \quad B_1 = \sum_{s=1}^{n} \frac{k_s^2 (1 - 3b_s) c_s}{k_1^2 (1 - 3b_1) + \gamma_1 \left([k_s^2] - 3 [k_s^2 b_s] \right)}.$$

We now evaluate the quantity D. We use the equilibrium integral equation for the medium, written in fluctuations

$$\int_V \sigma_{ij}' \varepsilon_{ij}' d\mathbf{r} = 0, \tag{1.11}$$

from which $D = \langle \sigma_{ij} \rangle \langle \varepsilon_{ij} \rangle$ follows. Hence eliminating the mean strain rates $\langle \varepsilon_{ij} \rangle$, and from (1.10) also, we find the macroscopic flow surface of the medium

$$\langle s_{ij} \rangle \langle s_{ij} \rangle + a^* \langle \sigma_{pp} \rangle^2 = k^{*2}. \tag{1.12}$$

Here

$$k^* = k_1 \sqrt{\frac{(1 - \alpha_1) A_1}{1 - \alpha_1 A_1}}, \quad a^* = \frac{(1 - \alpha_1) A_1 (1 - \gamma_1 B_1)}{3 (1 - \alpha_1 A_1)(1 - \gamma_1)(1 - 3b_1) B_1} \tag{1.13}$$

are the effective yield point and a parameter characterizing the macroscopic compressibility of the composite material.

Formulas for the effective constants k^*, a^* show that the model constructed corresponds to a composite medium for which the first component is the binding matrix while the remaining components are distributed therein as separate inclusions. Thus, if we set $k_1 = 0$, then k^* vanishes identically, but if $k_1 \neq 0$, then for any $k_s = 0$ ($s = 2, 3, \ldots, n$) the effective yield point is not zero identically. For $n = 2$ the formulas for k^* and a^* agree with the expressions for the effective constants of an ideally plastic two-phase medium obtained in [1].

As a particular case of the general formulas (1.13), we consider a porous medium whose plastically incompressible host material contains an absolutely rigid inclusion. Let V_1 be the volume of the host for which $a_1 = 0$, $\alpha_1 = 0.4$, $\gamma_1 = 1$; V_2 is the pore volume ($k_2 = 0$), V_3 is the volume of the rigid inclusions ($k_3 \to \infty$). Then $A_1 = c_1 + 2.5c_3$, $B_1 = c_1 + c_3$, and the flow surface for such a medium takes the form:

$$\langle s_{ij} \rangle \langle s_{ij} \rangle + \frac{c_2 (2 - 2c_2 + 3c_3)}{2 (1 - c_2)(6 + 4c_2 - 3c_3)} \langle \sigma_{pp} \rangle^2 = k_1^2 \frac{3 (2 - 2c_2 + 3c_3)}{6 + 4c_2 - 3c_3}. \tag{1.14}$$

For $c_3 = 0$ (no rigid phase), Eq. (1.14) agrees with the expression for the flow surface of a porous medium [1].

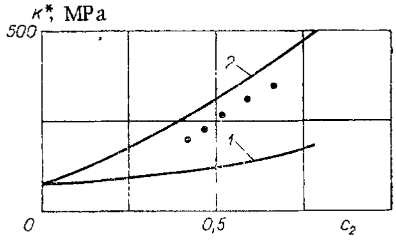

Fig. 1

2. In constructing and fabricating multicomponent composite materials, the binding host can be formed by several rather than just one component. For example, a material is obtained as a result of impregnating sintered tungsten powder by a copper melt, in which both phases form a splicing of interpenetrating skeletons. The formulas (1.13) cannot describe the behavior of such multicomponent media. Another method of taking the average of the system of equilibrium equations is required to estimate their mechanical properties [5].

We separate all the components of the composite materials into two groups by considering that the first m components possess an identical coherence and form a matrix of interpenetrating skeletons while the remaining n − m components are distributed in this matrix in the form of individual inclusions. Denoting the mean values of the constants in the matrix volume by

$$\{k^2\} = \sum_{s=1}^{m} k_s^2 c_s, \quad \{k^2 b\} = \sum_{s=1}^{m} k_s^2 b_s c_s,$$

we write the system of equations (1.13) in the form

$$\{k^2\}\varepsilon'_{ij,j} - \{k^2 b\}\varepsilon'_{pp,i} - \lambda'_{ij,j} = 0, \tag{2.1}$$

$$-\lambda_{ij} = \sum_{s=1}^{m} k_s^2 (\varepsilon_{ij} - \delta_{ij} b_s \varepsilon_{pp}) \varkappa'_s + \sum_{s=m+1}^{n} k_s^2 (\varepsilon_{ij} - \delta_{ij} b_s \varepsilon_{pp}) \varkappa_s.$$

By using the Green's tensor

$$G_{ik}(\mathbf{r}) = \frac{1}{8\pi\{k^2\}}\left(\delta_{ik} r_{,pp} - \frac{\{k^2\} - \{k^2 b\}}{2\{k^2\} - \{k^2 b\}} r_{,ik}\right)$$

the system (2.1) is reduced to a system of integral equations

$$\varepsilon'_{ij}(r) = \int_V G_{i(k,l)j}(\mathbf{r} - \mathbf{r}_1)\lambda'_{kl}(\mathbf{r}_1)\, d\mathbf{r}_1. \tag{2.2}$$

Evaluation of the quantity $\langle \varkappa'_s \varepsilon'_{ij} \rangle$ by using the singular approximation hypothesis in (2.2) yields

$$\langle \varkappa'_s \varepsilon'_{ij} \rangle = \left(\alpha \langle \varkappa'_s \lambda'_{ij} \rangle - \delta_{ij}\beta \langle \varkappa'_s \lambda'_{pp} \rangle\right)/\{k^2\}. \tag{2.3}$$

Here

$$\alpha = (2/15)\left(\{k^2\} - 3\{k^2 b\}\right)/\left(\{k^2\} - \{k^2 b\}\right),$$

$$\beta = (1/15)\left(\{k^2\} - 2\{k^2 b\}\right)/\left(\{k^2\} - \{k^2 b\}\right).$$

We determine the strain rate components averaged over the volume from (1.6) and (2.3)

$$\langle e_{ij} \rangle_s = \frac{\alpha D \langle s_{ij} \rangle + (1 - \alpha)\{k^2\}\langle e_{ij} \rangle}{\{k^2\}(1 + \alpha(m_s - 1))}, \tag{2.4}$$

$$\langle \varepsilon_{pp} \rangle_s = \frac{\gamma D \langle \sigma_{pp} \rangle + (1 - \gamma)\left(\{k^2\} - 3\{k^2 b\}\right)\langle \varepsilon_{pp} \rangle}{\left(\{k^2\} - 3\{k^2 b\}\right)(1 + \gamma(q_s - 1))}.$$

Here

$$m_s = k_s^2/\{k^2\}; \quad q_s = \left(k_s^2 - 3k_s^2 b_s\right)/\left(\{k^2\} - 3\{k^2 b\}\right);$$

$$\gamma = \left(\{k^2\} - 3\{k^2 b\}\right)/3\left(\{k^2\} - \{k^2 b\}\right).$$

Substituting (2.4) into (1.5) and extracting the deviator and volume parts result in the expressions

$$D\langle s_{ij}\rangle = \{k^2\}\frac{(1-\alpha)A}{\{k^2\}-\alpha A}\langle e_{ij}\rangle,$$
$$D\langle\sigma_{pp}\rangle = (\{k^2\}-3\{k^2b\})\frac{(1-\gamma)B}{\{k^2\}-3\{k^2b\}-\gamma B}\langle\varepsilon_{pp}\rangle. \quad (2.5)$$

Here

$$A=\sum_{s=1}^{n}\frac{k_s^2c_s}{1+\alpha(m_s-1)};\quad B=\sum_{s=1}^{n}\frac{k_s^2(1-3b_s)c_s}{1+\gamma(q_s-1)}.$$

The macroscopic flow surface of the medium is found by eliminating the quantities D and $\langle\varepsilon_{ij}\rangle$ from (1.11) and (2.5), and has the form (1.12). The effective constants of the composite material are determined from the formulas

$$k^* = \sqrt{\{k^2\}\frac{(1-\alpha)A}{\{k^2\}-\alpha A}},$$
$$a^* = \frac{(1-\alpha)\{k^2\}A(\{k^2\}-3\{k^2b\}-\gamma B)}{3(1-\gamma)(\{k^2\}-3\{k^2b\})B(\{k^2\}-\alpha A)}. \quad (2.6)$$

As a result the effective constants of a porous medium whose host material is formed by two plastic incompressible interpenetrating components can be obtained from the general formulas (2.6). Setting m = 2, n = 3, $a_1 = a_2 = 0$, $k_3 = 0$ into the relations (2.6), we find

$$k^* = k_1 v\sqrt{\frac{15p(3v+p^2q)}{5(2p+3v)(2pq+3v)-10v(3v+p^2q)}}, \quad (2.7)$$
$$a^* = k^{*2}(1-v)/6pk_1^2v^2,\quad q=k_2^2/k_1^2,\quad p=\frac{c_1+qc_2}{v},\quad v=1-c_3.$$

Let us apply (1.13) and (2.6) to a computation of the effective yield point of a two-phase medium whose components are plastically incompressible. Setting $a_1 = a_2 = 0$ into (1.13) and (2.6), we obtain the quantity $a^* \equiv 0$, from which there follows that both composites are macroscopically plastically incompressible. Moreover, the relationships

$$\alpha_1=\alpha=0{,}4,\quad \{k^2\}=\langle k^2\rangle=c_1k_1^2+c_2k_2^2,$$
$$m_s=k_s^2/(c_1k_1^2+c_2k_2^2),\quad A_1=c_1+\frac{5c_2k_2^2}{5k_1^2+2(k_2^2-k_1^2)},$$
$$A=\frac{5c_1k_1^2}{2+5(m_1-1)}+\frac{5c_2k_2^2}{2+5(m_2-1)}$$

hold. The macroscopic flow surface (1.12) takes the form

$$\langle s_{ij}\rangle\langle s_{ij}\rangle = k^{*2},$$

where

$$k^* = k_1\sqrt{1+\frac{5c_2(k_2^2-k_1^2)}{5k_1^2+2c_1(k_2^2-k_1^2)}} \quad (2.8)$$

is the effective yield point of the composite with individual inclusions, and

$$k^* = \sqrt{c_1k_1^2+c_2k_2^2-\frac{2c_1c_2(k_2^2-k_1^2)^2}{5(c_1k_1^2+c_2k_2^2)+2(c_1-c_2)(k_2^2-k_1^2)}} \quad (2.9)$$

is the effective yield point of a composite material formed by interpenetrating skeletons.

Presented in the figure is a comparison between (2.8) and (2.9) and experimental measurement results for the conditional yield point of a copper matrix-sintered tungsten powder composite [6]. Curve 1 corresponds to (2.8) and 2 to (2.9). The experimental values are displayed by points.

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